

Scattering and Bound State Green's Functions on a Plane via $so(2,1)$ Lie Algebra

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Abstract

We calculate the Green's functions for the particle-vortex system, for two anyons on a plane with and without a harmonic regulator and in a uniform magnetic field. These Green's functions which describe scattering or bound states (depending on the specific potential in each case) are obtained exactly using an algebraic method related to the $SO(2,1)$ Lie group. From these Green's functions we obtain the corresponding wave functions and for the bound states we also find the energy spectra.

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1 Introduction

In this paper we study exactly solvable problems for one or two particles on a plane bound or not by external potentials by constructing algebraically their Green's functions. These Green's functions describe scattering or bound states, depending on the kind of interaction in each situation. The algebraic method used here is based on the Schwinger representation [1] for the inverse of an operator which is an integral representation involving the exponential of the operator. The operator is identified with the hamiltonian of the problem which can be written as a linear combination of the generators of a Lie algebra. In particular, we are interested in the $so(2,1)$ Lie algebra which describe some well known problems as the harmonic oscillator, the hydrogen atom and the Morse potential [2]-[16]. Some other problems with more involved potentials can also be described by this algebra (see, for instance refs. [17]-[32]). Once the hamiltonian is written in terms of the $so(2,1)$ generators one can use Baker-Campbell-Hausdorff (BCH) formulas [33]-[35] to split the exponential of the hamiltonian into a convenient product of the $so(2,1)$ generators. BCH formulas are also used to change the order of the product of the exponentials of generators to simplify the computation of the Green's functions. This method was used to describe the Dirac electron in a Coulomb potential [9] and the discussion presented here is a non-relativistic version modified to include other potentials [16, 27]. As we will see, for the simplest cases the hamiltonian is identified simply with just *one* $so(2,1)$ generator. In these particular cases just one BCH formula is used.

The two dimensional problems we are going to discuss here have been studied before in [36]-[48] with other approaches, although for instance, in [43] the $so(2,1)$ symmetry was invoked to construct the wave function for the particle-vortex system. The problem of particles moving on a plane is relevant to the studies of condensed matter systems as the fractional quantum Hall effect and anyonic superconductivity [49]-[57], supersymmetry [58, 59], and fault-tolerant quantum computing [60]. Free anyon Green's functions have been recently used to study correlation functions of anyon interferometry [61].

This paper is organized as follows: In section 2 we discuss the particle-vortex system and its $so(2,1)$ dynamical algebra and in section 3 we use it obtain its Green's function algebraically. From this Green's function we obtain the wave functions and find a continuous energy spectrum. In section 4, we discuss the two anyon problem on a plane

within a harmonic well and obtain its Green's function using the above mentioned algebraic method related to the $SO(2,1)$ Lie group. From this Green's function we obtain the corresponding wave functions and the discrete energy spectrum. Then, in section 5, we obtain the Green's function for the problem of two anyons without any regulator from the results of the previous section. We also show that this problem is equivalent to the particle-vortex system, if one identifies the quantized flux of the particle-vortex with the anyon statistical parameter. Finally, in section 6 we obtain exactly the Green's function for two anyons in a uniform magnetic field and the corresponding wave functions and the discrete energy spectrum. In section 7 we present our conclusions.

2 The particle-vortex system

Let us start the discussion with the particle-magnetic vortex which is defined to be a two-dimensional system characterized by the Schrödinger equation

$$i\hbar \frac{\partial \Psi}{\partial t} = H\Psi. \quad (1)$$

The hamiltonian of interest is given by

$$H = \frac{1}{2M} \left(\vec{p} - \frac{e}{c} \vec{A} \right)^2 \quad (2)$$

with an externally prescribed vector potential

$$\vec{A} = \frac{\Phi}{2\pi r^2} \mathbf{e}_3 \wedge \vec{r} \quad (3)$$

where \mathbf{e}_3 is a constant unit vector perpendicular to the plane in which \vec{r} lies.

The vector potential gives rise to a magnetic field

$$\begin{aligned} B &= \nabla \wedge \vec{A} \\ &= \Phi \delta^2(\vec{r}) \end{aligned} \quad (4)$$

with flux $\Phi = \int B(\vec{r}) d^2r$.

A simple example is given by the motion of a charged particle around a magnetic flux line when the force motion parallel to the flux line is ignored. If one solves the two dimensional problem one may apply a boost along the direction of the flux line and get

the description of the three dimensional system. The experimental set-up was considered in the Bohm-Aharonov effect [36].

Clearly, the system is invariant under two dimensional rotations whose generator is given by the conserved angular momentum

$$J = \vec{r} \wedge \vec{p} \quad (5)$$

where \vec{p} is the canonical momentum and J generates the $O(2)$ group of rotations in a plane. Also, since H does not have an explicit time dependence it is a constant of motion. In principle we expect to have two more constants of motion. In an interesting paper Jackiw [43] constructed them in explicit form as

$$D = t H - \frac{1}{4}(\vec{r} \cdot \vec{p} + \vec{p} \cdot \vec{r}) \quad (6)$$

$$K = -t^2 H + 2tD + \frac{Mr^2}{2} \quad (7)$$

where the fact that $\vec{r} \cdot \vec{A} = 0$ in (6) was used.

It may be noted that D and K are generators of scale and conformal transformations which change the Lagrangian by a total time derivative. One can verify that

$$[H, D] = -i\hbar H \quad (8)$$

$$[D, K] = -i\hbar K \quad (9)$$

$$[K, H] = +2i\hbar D \quad (10)$$

which are the commutation relations of the generators of the algebra associated with the group $SO(2,1)$.

Since J commutes with H , D and K the symmetry group of the system is the direct product $SO(2) \times SO(2,1)$. In the following we consider the construction of the Green's function for the particle vortex system by making use of the Baker-Campbell-Hausdorff (BCH) formulas for the exponentials of the generators of the algebra of $SO(2,1)$ group. Since we separate the time variable we only need the form of the generators at $t = 0$.

The Green's function associated with the particle-vortex system satisfies the equation

$$\left(i\hbar \frac{\partial}{\partial t} - H \right) G(\vec{r}, t; \vec{r}', t') = -\delta(\vec{r} - \vec{r}') \delta(t - t'). \quad (11)$$

Since H is time independent one may write

$$G(\vec{r}, t; \vec{r}', t') = \frac{1}{2\pi} \int dE G_E(\vec{r}, \vec{r}') e^{-iE(t-t')/\hbar} \quad (12)$$

and get

$$(-H + E) G_E(\vec{r}, \vec{r}') = \delta(\vec{r} - \vec{r}'). \quad (13)$$

Then, using the Schwinger [1] representation for the inverse of an operator one can write the Green's function as

$$G_E(\vec{r}, \vec{r}') = \frac{i}{\hbar} \int_0^\infty ds e^{i(E-H+i\epsilon)s/\hbar} \delta(\vec{r} - \vec{r}'), \quad (14)$$

where s is usually known as the Schwinger's proper time and $\epsilon > 0$ is included to assure the convergence of the above integral. Let us now calculate explicitly this Green's function algebraically.

3 Green's function for the particle-vortex system

Since H is invariant under rotations we may use the result

$$\delta(\vec{r} - \vec{r}') = \frac{1}{r} \delta(r - r') \delta(\phi - \phi'), \quad (15)$$

where

$$\delta(\phi - \phi') = \frac{1}{2\pi} \sum_m e^{im(\phi - \phi')}, \quad (16)$$

with integer m . Thus if we write

$$G_E(\vec{r}, \vec{r}') = \frac{1}{2\pi} \sum_m e^{im(\phi - \phi')} G_{Em}(r, r') \quad (17)$$

then the one dimensional Green's function is

$$G_{Em}(r, r') = \frac{i}{\hbar r'} \int_0^\infty ds e^{i(E-H_m)s/\hbar} \delta(r - r') ds, \quad (18)$$

where

$$H_m = -\frac{\hbar^2}{2M} \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} - \frac{(m - \nu)^2}{r^2} \right) \quad (19)$$

is the radial Hamiltonian dependent on the integer angular momentum quantum number m and ν is the quantized flux:

$$\nu = \frac{e\Phi}{2\pi\hbar c}. \quad (20)$$

In the following we use the set of differential operators given by

$$T_1 = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} - \frac{(m - \nu)^2}{r^2} \quad (21)$$

$$T_2 = -\frac{i}{2} \left(r \frac{\partial}{\partial r} + 1 \right) \quad (22)$$

$$T_3 = -\frac{1}{8} r^2. \quad (23)$$

which can be easily related to the ones defined in the previous section. These operators satisfy the $so(2, 1)$ Lie algebra

$$[T_1, T_2] = -iT_1 \quad (24)$$

$$[T_2, T_3] = -iT_3 \quad (25)$$

$$[T_3, T_1] = +iT_2, \quad (26)$$

as well as the operators H, K, D . Next we note the following representation for the delta function

$$\delta(r - r') = \frac{M}{4\pi i r'^{\delta-1}} \int_{-\infty}^{i\infty} e^{\frac{1}{4} q M (r^2 - r'^2)} r'^{\delta} dq, \quad (27)$$

where $r, r' \geq 0$, and the arbitrary parameter δ will be fixed later. Thus the one dimensional Green's function can be written as

$$G_{Em}(r, r') = \frac{M}{4\pi\hbar r'^{\delta}} \int_0^{\infty} e^{iEs/\hbar} ds \int_{-\infty}^{i\infty} e^{-\frac{qM}{4} r'^2} e^{\frac{is\hbar}{2M} T_1} e^{-2MqT_3} r'^{\delta} dq. \quad (28)$$

Further, one can show that the following identity holds

$$e^{\frac{is\hbar}{2M} T_1} e^{-2MqT_3} = e^{-i\zeta_3 T_3} e^{-i\zeta_2 T_2} e^{-i\zeta_1 T_1} \quad (29)$$

where

$$e^{\zeta_2/2} = \frac{s\hbar}{2i} \left(q + \frac{2i}{s\hbar} \right) \quad (30)$$

$$-i\zeta_3 = -2M \left(\frac{2i}{s\hbar} + \frac{4}{(s\hbar)^2 \left(q + \frac{2i}{s\hbar} \right)} \right). \quad (31)$$

The proof of the above relations is given, e. g., in [16, 27] where ζ_1 is also calculated. Note however that the value of ζ_1 is not needed here since we will choose δ such that

$$T_1 r^\delta = 0 \quad (32)$$

which implies

$$\delta = |m - \nu|. \quad (33)$$

In fact the condition (32) also allows $\delta = -|m - \nu|$, but these negative values lead to unphysical solutions as will be seen in the following. Next, it is easy to verify that

$$e^{-i b T_2} f(r) = e^{-b/2} f(r e^{-b/2}) \quad (34)$$

so that

$$\begin{aligned} G_{Em}(r, r') &= \frac{M r^\delta}{4\pi \hbar r'^\delta} \int_0^\infty ds \exp\left\{\frac{i E s}{\hbar}\right\} \exp\left\{\frac{i M}{2s\hbar} r^2\right\} \int_{-\infty}^{\infty} dq \exp\left\{-\frac{q M}{4} r'^2\right\} \\ &\times \exp\left\{\frac{M r^2}{(s\hbar)^2(q + \frac{2i}{s\hbar})}\right\} \left[\frac{s\hbar}{2i} \left(q + \frac{2i}{s\hbar}\right)\right]^{-(1+\delta)}. \end{aligned} \quad (35)$$

Expanding the exponential as a power series and doing the integrations by repeatedly using the result

$$\frac{1}{2\pi i} \int_{-\infty}^{\infty} dq \frac{\exp\{-\frac{q M}{4} r'^2\}}{\left(q + \frac{2i}{s\hbar}\right)^{\xi+1}} = \frac{\exp\{\frac{i M}{2s\hbar} r'^2\} \left(-\frac{M}{4} r'^2\right)^\xi}{\Gamma(\xi + 1)}, \quad (36)$$

we find that the one dimensional Green's function can be written as

$$G_{Em}(r, r') = -\frac{M}{\hbar} \int_0^\infty \frac{ds}{s} e^{i E s/\hbar} \exp\left\{\frac{i M}{2s\hbar} (r^2 + r'^2)\right\} e^{-\frac{i\pi}{2}\delta} J_\delta\left(\frac{M r r'}{s\hbar}\right), \quad (37)$$

where we have used the definition of Bessel functions

$$J_\delta(z) = \left(\frac{z}{2}\right)^\delta \sum_n \frac{(-z^2/4)^n}{n! \Gamma(n + \delta + 1)}. \quad (38)$$

Next using the result [62]

$$\int_0^\infty dz e^{-\xi z} J_\delta(2\beta\sqrt{z}) J_\delta(2\gamma\sqrt{z}) = \frac{1}{\xi} e^{-\frac{1}{\xi}(\beta^2 + \gamma^2)} I_\delta\left(\frac{2\beta\gamma}{\xi}\right) \quad (39)$$

valid for $\Re(\delta) > -1$ and identifying $I_\delta(z) = i^{-\delta} J_\delta(iz)$ we get

$$G_{Em}(r, r') = \int_0^\infty dE' \frac{\mathcal{U}_{E'}^m(r) \mathcal{U}_{E'}^m(r')}{E - E' + i\epsilon}, \quad (40)$$

where

$$\mathcal{U}_{E'}^m(r) = \frac{\sqrt{M}}{\hbar} J_\delta(\sqrt{2ME'} \frac{r}{\hbar}). \quad (41)$$

Substituting this result into equation (17) we get

$$G_E(\vec{r}, \vec{r}') = \int_0^\infty dE' \frac{1}{E - E' + i\epsilon} \sum_m \frac{e^{im(\phi - \phi')}}{2\pi} \mathcal{U}_{E'}^m(r) \mathcal{U}_{E'}^m(r') \quad (42)$$

Note that eq. (40) is the spectral representation of the one dimensional Green's function (18), $\mathcal{U}_{E'}^m(r)$ are the corresponding one dimensional wave functions and the energy spectrum is continuous for energy $E > 0$ and angular momentum m , in agreement with [43]. Note that this should be the case since the Hamiltonian (19) corresponds to a particle in a nonconfining potential of a centrifugal barrier $1/r^2$.

It is interesting to note that the above calculation leading to the wave functions (41) is rather different from Jackiw's [43] group theoretical discussion. To trace the main differences first we mention that he considered the operators

$$R = \frac{1}{2} \left(\frac{1}{a} K + aH \right) \quad (43)$$

$$S = \frac{1}{2} \left(\frac{1}{a} K - aH \right) \quad (44)$$

where a is a fixed parameter with time dimensionality. The operators R , S and D also close the $so(2,1)$ Lie algebra, analogous to (8)-(10). Then, he calculated the eigenstates of R . Note that for any fixed time, as for instance $t = 0$, the operator

$$R = \frac{1}{2} \left(\frac{1}{a} \frac{Mr^2}{2} + aH \right) \quad (45)$$

has a *discrete* spectrum because of the presence of the bounding potential r^2 . Next, using further group theoretical methods he expressed the *continuous* eigenstates of H in terms of the eigenstates of R . This procedure can be understood as an infrared cutoff regularization of the continuous eigenstates of H . Our results are in agreement with those present by Jackiw [43]. In next section we are going to discuss the problem of anyons in a

harmonic well which, as we will discuss, can be interpreted as the particle-vortex problem with a harmonic regulator.

Before we move to next section, let us comment that the Green's function (42) can also be written in terms of associated Laguerre's polynomials $L_n^\xi(x)$. Using the relation

$$J_\xi(2\sqrt{xz}) = e^{-z}(xz)^{\xi/2} \sum_{n=0}^{\infty} \frac{z^n L_n^\xi(x)}{\Gamma(n + \xi + 1)} \quad (46)$$

valid for $\xi > -1$. Identifying $x = Mr^2/\hbar$, $z = E/2\hbar$ we get

$$\mathcal{U}_{E'}^m(r) = \frac{\sqrt{M}}{\hbar} e^{-E'/2\hbar} \left(\frac{E' Mr^2}{2\hbar^2} \right)^{\delta/2} \sum_{n=0}^{\infty} \left(\frac{E'}{2\hbar} \right)^n \frac{L_n^\delta(\frac{Mr^2}{\hbar})}{\Gamma(n + \delta + 1)}, \quad (47)$$

so that

$$\begin{aligned} G_E(\vec{r}, \vec{r}') &= \frac{M}{2\pi\hbar} \int_0^\infty \frac{dE' e^{-E'/\hbar}}{E - E' + i\epsilon} \sum_m e^{im(\phi - \phi')} \left(\frac{E' Mr r'}{2\hbar^2} \right)^\delta \\ &\times \sum_{n=0}^{\infty} \left(\frac{E'}{2\hbar} \right)^n \frac{L_n^\delta(\frac{Mr^2}{\hbar})}{\Gamma(n + \delta + 1)} \sum_{\ell=0}^{\infty} \left(\frac{E'}{2\hbar} \right)^\ell \frac{L_\ell^\delta(\frac{Mr'^2}{\hbar})}{\Gamma(\ell + \delta + 1)}. \end{aligned} \quad (48)$$

Further, one can also perform the integration in E' so that this Green's function can be rewritten as

$$\begin{aligned} G_E(\vec{r}, \vec{r}') &= -\frac{M}{2\pi\hbar} e^{-E/\hbar} \sum_m e^{im(\phi - \phi')} \left(\frac{Mr r'}{\hbar} \right)^\delta \sum_{n=0}^{\infty} \sum_{\ell=0}^{\infty} L_n^\delta \left(\frac{Mr^2}{\hbar} \right) L_\ell^\delta \left(\frac{Mr'^2}{\hbar} \right) \\ &\times (-E)^{n+\ell+\delta} \frac{\Gamma(n + \ell + \delta + 1)}{\Gamma(n + \delta + 1)\Gamma(\ell + \delta + 1)} \Gamma \left(-n - \ell - \delta, -\frac{E}{\hbar} \right) \end{aligned} \quad (49)$$

where $\Gamma(a, x)$ is the incomplete gamma function.

4 Two anyons in a harmonic well

Anyons are quasi-particles that obey fractional statistics, i. e., an intermediate statistics between the Bose-Einstein and Fermi-Dirac cases [37]-[41]. The boson and fermion wave functions differ by the interchange of two or more identical particles. The bosonic wave function is completely symmetric while the fermionic is completely antisymmetric. Then, the two-particle wave functions before and after the interchange of two particles are related by:

$$\psi(\vec{r}_2, \vec{r}_1) = e^{i\theta} \psi(\vec{r}_1, \vec{r}_2) \quad (50)$$

where θ determines the statistics of the system. If $\theta = 0$ (modulo 2π) the system obeys Bose-Einstein statistics and if $\theta = \pi$ (modulo 2π) the system obeys Fermi-Dirac statistics. In three spatial dimensions these are the only allowed possibilities. However, if the particles are restricted to live in two spatial dimensions θ can assume any real value interpolating the BE and FD statistics.

Considering the path integral formulation of quantum mechanics, the transition amplitude between two states is proportional to $\exp\{iS/\hbar\}$, where S is the classical action. Then, to reproduce the above behavior we consider that a two anyon system (any θ) can be represented by a conventional lagrangean L plus a topological term [57]:

$$L \rightarrow L_\theta = L + \frac{\theta}{\pi} \dot{\phi} \quad (51)$$

where ϕ is the relative angle between \vec{r}_1 and \vec{r}_2 . This way, turning around one particle in respect to the other by an angle $\phi = \pi$ we obtain a phase:

$$\exp \left\{ \frac{i\theta}{\pi} \int_0^\pi d\phi \right\} = \exp \{i\alpha\pi\}, \quad (52)$$

where

$$\alpha = \frac{\theta}{\pi}, \quad (53)$$

is the statistical parameter of the two anyon system.

Here, in this section we consider a two anyon system characterized by the coordinates \vec{r}_1 and \vec{r}_2 moving on a plane subjected to a harmonic regulator $V(\vec{r}_1, \vec{r}_2) = \frac{1}{2}m_0\omega^2(r_1^2 + r_2^2)$, where m_0 is the mass of each anyon. The use of a potential as a regulator is not mandatory but it is usual in the literature [41] since it simplifies the discussion once the spectrum becomes discrete. An alternative regulator procedure is to use boundary conditions as considered in Arovas et al [40] to calculate the second virial coefficient of the two anyon system. Nonetheless, it is also possible to avoid the use of any regulator and consider the case of “free” anyons, as will be discussed in the next section. This situation is in fact related to the case of the magnetic vortex discussed in the previous section. The connection of these cases will be discussed in the next section.

Introducing the center of mass and relative coordinates $\vec{R} = \vec{r}_1 + \vec{r}_2$, and $\vec{r} = \vec{r}_1 - \vec{r}_2 = (r, \phi)$, respectively, the lagrangean for the two anyon system with a harmonic regulator

can be written as:

$$L_\alpha = \frac{1}{2}M\dot{R}^2 + \frac{1}{2}M\omega^2 R^2 + \frac{1}{2}\mu(\dot{r}^2 + r^2\dot{\phi}^2 + \omega^2 r^2) + \alpha\hbar\dot{\phi}, \quad (54)$$

where $M = 2m_0$, $\mu = m_0/2$. The motion of the center of mass is described by the first part of the lagrangean corresponding to a two dimensional harmonic motion that does not contribute to the statistical behavior. From now on we will only consider the relative motion of the two anyon system. The canonical momenta are then given by:

$$p_\phi = \frac{\partial L}{\partial \dot{\phi}} = \mu r^2 \dot{\phi} + \alpha\hbar; \quad p_r = \mu \dot{r}, \quad (55)$$

and the Hamiltonian of the relative motion of the two particles is

$$H = -\frac{\hbar^2}{2\mu} \left[\frac{1}{r} \frac{\partial}{\partial r} r \frac{\partial}{\partial r} + \frac{1}{r^2} \left(\frac{\partial}{\partial \phi} - i\alpha \right)^2 \right] + \frac{1}{2}\mu\omega^2 r^2. \quad (56)$$

Here we are going to construct the Green's function for this problem using an algebraic method associated with the dynamical $SO(2,1)$ Lie group [16, 27]. The technique used here is a generalization of the one presented in the previous section, and we will recover that results in the following sections. The Green's functions for this problem satisfies the equation

$$(H - E)G_E(r, r', \phi, \phi') = \frac{\delta(r - r')}{r} \delta(\phi - \phi') \quad (57)$$

where $G_E(\vec{r}, \vec{r}') \equiv G_E(r, r', \phi, \phi')$. Decomposing this Green's function as we did in the previous section, eq. (17), one can obtain the radial hamiltonian H_r for this problem and we write the resolvent operator as:

$$\Lambda_r = (H_r - E) = g_0 + g_1 T_1(r) + g_3 T_3(r) \quad (58)$$

where the operators T_i are given by eqs. (21)-(23) which satisfy the $so(2,1)$ Lie algebra eqs. (24-26) and the parameters g_i are given by

$$g_0 = -E; \quad g_1 = -\frac{\hbar^2}{2\mu}; \quad g_3 = -4\mu\omega^2. \quad (59)$$

Note that $H_r = H_m + \frac{1}{2}\mu\omega^2 r^2$, with H_m given by eq. (19), if we further identify α with ν . Using the Schwinger [1] representation as before we find:

$$G_E(r, r') = i \int_0^\infty ds \exp[-is(g_0 + g_1 T_1(r) + g_3 T_3(r) - i\epsilon)] \frac{\delta(r - r')}{r} \quad (60)$$

In addition to eq. (29), here we need another Baker-Campbell-Hausdorff formula

$$\exp \left\{ -i \frac{s}{\hbar} (g_1 T_1 + g_3 T_3) \right\} = \exp \{ -ia T_3 \} \exp \{ -ib T_2 \} \exp \{ -ic T_1 \} \quad (61)$$

where the parameters a , b , c , and k are given by:

$$a = 2 \frac{k}{g_1} \tan k \frac{s}{\hbar}, \quad b = 2 \ln \left(\cos k \frac{s}{\hbar} \right), \quad c = \frac{g_1}{k} \tan k \frac{s}{\hbar} \quad k = \sqrt{\frac{g_1 g_3}{2}}. \quad (62)$$

Following ref. [16] we find

$$\begin{aligned} & \exp \left\{ -i \frac{s}{\hbar} (g_1 T_1 + g_3 T_3) \right\} \frac{\delta(r - r')}{r} \\ &= -\frac{ik \exp \{ 2i\pi\delta \}}{2g_1 \sin(ks/\hbar)} r' \exp \left\{ -\frac{ik}{4g_1} (r'^2 + r^2) \cot k \frac{s}{\hbar} \right\} I_\delta \left(-\frac{k r' r}{2ig_1 \sin(ks/\hbar)} \right) \end{aligned} \quad (63)$$

with δ given by:

$$\delta = |m - \alpha|. \quad (64)$$

As before, $\delta = -|m - \alpha|$ would imply non-normalizable solutions and we will not consider this case.

Then we obtain the Green's function as:

$$\begin{aligned} G_E(r, r') &= -\frac{i}{\hbar} \int_0^\infty ds \exp \left\{ -i \frac{s}{\hbar} (g_0 - i\epsilon) \right\} \frac{ik \exp(2i\pi\delta)}{2g_1 \sin(ks/\hbar)} (r r')^{\frac{1}{2}} \\ &\times \exp \left\{ -\frac{ik}{4g_1} (r^2 + r'^2) \cot(ks/\hbar) \right\} I_\delta \left[\frac{-k r r'}{2ig_1 \sin(ks/\hbar)} \right] \end{aligned} \quad (65)$$

where I_δ is the modified Bessel function of order δ . This Bessel function is related to the associated Laguerre's polynomials L_n^δ by:

$$I_\delta \left(2 \frac{\sqrt{y' y z}}{1 - z} \right) \exp \left\{ -z \frac{y' + y}{1 - z} \right\} = (y' y z)^{\delta/2} (1 - z) \sum_{n=0}^{+\infty} \frac{n!}{\Gamma(n + \delta + 1)} L_n^\delta(y) L_n^\delta(y') z^n. \quad (66)$$

Then integrating over the proper time s one finds for the Green's function of two anyons on a plane with a harmonic regulator:

$$\begin{aligned} G_E(r, r', \phi, \phi') &= -\frac{1}{\pi} \sum_{m=-\infty}^{+\infty} e^{2\pi i |m - \alpha|} \left(\frac{\mu\omega}{\hbar} \right)^{1+|m-\alpha|} (r r')^{|m-\alpha|} \\ &\times e^{im(\phi - \phi')} \sum_{n=0}^{\infty} \frac{n! L_n^{|m-\alpha|} \left(\frac{\mu\omega}{\hbar} r^2 \right) L_n^{|m-\alpha|} \left(\frac{\mu\omega}{\hbar} (r')^2 \right)}{\Gamma(n + |m - \alpha| + 1)} \\ &\times \exp \left\{ -\frac{\mu\omega}{2\hbar} (r^2 + r'^2) \right\} \frac{1}{E - \hbar\omega(2n + |m - \alpha| + 1)}. \end{aligned} \quad (67)$$

If one writes the spectral representation for the Green's function as:¹

$$G(r, r', \phi, \phi') = \sum_{n=0}^{\infty} \sum_{m=-\infty}^{\infty} \frac{\psi_{n,m}(r, \phi) \psi_{n,m}(r', \phi')}{E - E_{n,m}}, \quad (68)$$

so that the wave functions are given by its residues

$$\begin{aligned} \psi_{n,m}^{\alpha}(r, \phi) &= \frac{i}{\sqrt{\pi}} e^{\pi i |m-\alpha|} e^{im\phi} \left(\frac{\mu\omega}{\hbar} \right)^{\frac{1}{2}(1+|m-\alpha|)} r^{|m-\alpha|} \\ &\times \frac{\sqrt{n!} L_n^{|m-\alpha|} \left(\frac{\mu\omega}{\hbar} r^2 \right)}{\sqrt{\Gamma(n + |m-\alpha| + 1)}} \exp \left\{ -\frac{\mu\omega}{2\hbar} r^2 \right\}. \end{aligned} \quad (69)$$

and the poles correspond to the energy spectrum

$$E_{nm}^{\alpha} = \hbar\omega(2n + |m - \alpha| + 1) \quad (70)$$

where $n = 0, 1, 2, 3, \dots$ is the radial quantum number. This spectrum corresponds to that of a two dimensional harmonic oscillator with angular momentum $|m - \alpha|$.

If we had started with particles identified with bosons then the allowed angular momentum values would be $m = 0, \pm 2, \pm 4, \dots$. Had we started with fermions then the values of the angular momentum should be $m = \pm 1, \pm 3, \pm 5, \dots$. The quantized energy levels are periodic functions of the statistical parameter α with period 2, although the energy of a single state with quantum numbers (n, m) is not periodic. These conclusions are in agreement with [40, 41] where the second virial coefficient for anyons has been calculated without a harmonic regulator, but considering boundary conditions on the wave functions.

5 Two “free” anyons

In this section we are going to obtain the Green's function for two anyons as discussed above but without any regulator. Here, the relative motion Hamiltonian after separating the angular variable ϕ is

$$H = -\frac{\hbar^2}{2M} \left(\frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} - \frac{1}{r^2} (m - \alpha)^2 \right) \quad (71)$$

¹Note that the obtained Green's function is real up to a complex phase $i\pi|m - \alpha|$. Then we are using a nonstandard definition for the spectral decomposition to preserve this nontrivial phase which is related to the statistics of the system.

where $m = 0, 1, 2, \dots$ are again the eigenvalues of the angular momentum and $\alpha = \theta/\pi$. The $\text{so}(2,1)$ Lie algebra here is analogous to the one discussed in the previous section with generators defined by (21)-(23). Here the resolvent operator is simply

$$(H - E) = g_0 + g_1 T_1(r), \quad (72)$$

with g_0 and g_1 given by eq. (59) (here $g_3 = 0$), so that the radial Green's function can be written as:

$$G(r, r') = \frac{i}{\hbar} \int_0^\infty ds \exp\left\{-i\frac{s}{\hbar}(g_0 - i\epsilon)\right\} \exp\left\{-i\frac{s}{\hbar}g_1 T_1\right\} \frac{1}{r} \delta(r - r') \quad (73)$$

It has been shown in ref. [16] that:

$$\exp\{icT_1\} \frac{\delta(r - r')}{r} = i(-i)^{|m-\alpha|} \frac{M}{s\hbar} \exp\left\{\frac{iM}{2s\hbar}(r^2 - r'^2)\right\} J_{|m-\alpha|}\left(\frac{M}{s\hbar}rr'\right) \quad (74)$$

so that the radial Green's function is given by:

$$G(r, r') = -\frac{M}{\hbar} (-i)^{|m-\alpha|} \int_0^\infty \frac{ds}{s} \exp\left\{i\frac{s}{\hbar}(E + i\epsilon)\right\} \exp\left\{\frac{iM}{2s\hbar}(r^2 - r'^2)\right\} J_{|m-\alpha|}\left(\frac{M}{s\hbar}rr'\right). \quad (75)$$

Another way to approach the two anyon system without a regulator potential is to consider the two anyon system in the harmonic well, eq. (56), and take the limit where the regulator vanishes $\omega \rightarrow 0$. This limit corresponds to take $k \rightarrow 0$ in eq. (65), so that the above Green's function is reobtained.

This Green's function can be compared with the one obtained for the particle-vortex system, eq. (37). One can note that they are identical if one identifies the quantized flux ν , eq. (20), with the anyon statistical parameter α , eq. (53).

6 Two anyons in a uniform magnetic field

The Hamiltonian of the relative motion of two anyons in a uniform and constant magnetic field B is given by [42]-[48]

$$H = \frac{1}{2\mu} \left(\vec{p} + \frac{1}{2} \mu \omega_c r \hat{\phi} + \frac{\alpha \hbar}{r} \hat{\phi} \right)^2 \quad (76)$$

where $\omega_c = eB/mc$ is the cyclotron frequency, the second term on brackets corresponds to the physical (external) magnetic vector potential $\vec{A} = Br\hat{\phi}/2$ and the third term is the

statistical vector potential. This statistical term can be absorbed in the angular part of the kinetical term that contributes to the angular momentum of the particles. The radial part of this hamiltonian can be written as before as:

$$H = -\frac{\hbar^2}{2\mu} \left[\frac{1}{r} \frac{\partial}{\partial r} r \frac{\partial}{\partial r} - \frac{1}{r^2} (m - \alpha)^2 \right] - \frac{m\hbar\omega_c}{4} + \frac{1}{8}\mu\omega_c^2 r^2. \quad (77)$$

The presence of the magnetic field B implies an r -independent term that contributes to the energy but the form of this Hamiltonian is similar to that of the problem of two anyons in a harmonic well, discussed in section 4. This fact allows the use of the algebraic method as before to calculate the Green's functions. Then the $\text{so}(2,1)$ generators that describe the two anyons in a magnetic field are the ones given by eqs. (21)-(23). The resolvent operator is again given by eq. (58) with the parameters

$$g_0 = -\left(E + \frac{1}{4}m\hbar\omega_c\right) ; \quad g_1 = -\frac{\hbar^2}{2\mu} ; \quad g_3 = -\mu\omega_c^2. \quad (78)$$

Following the algebraic method as in section 4 with these parameters we find the Green's function:

$$\begin{aligned} G_E(r, r', \phi, \phi') &= -\frac{1}{\pi} \sum_{m=-\infty}^{+\infty} e^{2\pi i|m-\alpha|} \left(\frac{\mu\omega_c}{2\hbar}\right)^{1+|m-\alpha|} (rr')^{|m-\alpha|} e^{-im(\phi-\phi')} \\ &\times \sum_{n=0}^{\infty} \frac{n! L_n^{|m-\alpha|} \left(\frac{\mu\omega_c}{2\hbar} r^2\right) L_n^{|m-\alpha|} \left(\frac{\mu\omega_c}{2\hbar} (r')^2\right)}{\Gamma(n + |m - \alpha| + 1)} \\ &\times \exp\left\{-\frac{\mu\omega_c}{4\hbar}(r^2 + r'^2)\right\} \frac{1}{E - \frac{\hbar\omega_c}{2}(2n + |m - \alpha| + 1 + \frac{m}{2})}. \end{aligned} \quad (79)$$

From this Green's function we obtain the normalized wave functions:

$$\begin{aligned} \psi_{n,m}^\alpha(r, \phi) &= \frac{i}{\sqrt{\pi}} e^{i\pi|m-\alpha|} e^{-im\phi} \left(\frac{\mu\omega_c}{2\hbar}\right)^{\frac{1}{2}(1+|m-\alpha|)} r^{|m-\alpha|} \\ &\times \frac{\sqrt{n!} L_n^{|m-\alpha|} \left(\frac{\mu\omega_c}{2\hbar} r^2\right)}{\sqrt{\Gamma(n + |m - \alpha| + 1)}} \exp\left\{-\frac{\mu\omega_c}{4\hbar} r^2\right\}. \end{aligned} \quad (80)$$

and the corresponding energy levels:

$$E_{nm}^\alpha = \frac{\hbar\omega_c}{2} \left(2n + |m - \alpha| + 1 + \frac{m}{2}\right). \quad (81)$$

These energy levels coincide with the Landau levels if the anyon statistical contribution vanishes ($m = \alpha = 0$). In particular, the ground state wave function is obtained when

one takes $m = n = 0$:

$$\psi_{0,0}^\alpha(r) = \frac{i}{\sqrt{\pi}} e^{i\pi\alpha} \left(\frac{\mu\omega_c}{2\hbar} \right)^{\frac{1}{2}(1+\alpha)} \frac{r^\alpha}{\sqrt{\Gamma(1+\alpha)}} \exp \left\{ -\frac{\mu\omega_c}{4\hbar} r^2 \right\}. \quad (82)$$

7 Conclusions

In this paper we have calculated algebraically the Green's functions for one and two particles confined on a plane, namely the particle-vortex system and a pair of anyons with and without external potentials. The external potentials considered were a harmonic well and a uniform magnetic field. In these problems we have identified the hamiltonian operator in each case with the generators of the $SO(2,1)$ Lie group satisfying the $so(2,1)$ Lie algebra. From these algebraic properties we obtained all relevant dynamical quantities of each system. This means that these systems are described by the $so(2,1)$ dynamical algebra.

In particular, we calculated the Green's function for particle-vortex system and the Green's function for a pair of free anyons and found that these Green's functions are equivalent, once one identifies the quantized flux ν of the particle-vortex system with the anyon statistical parameter α . These two Green's functions exhibit respectively the phase factors $(-i)^{|m-\nu|}$ and $(-i)^{|m-\alpha|}$ as a common signature of fractional statistics for both systems.

We obtained also the Green's function for the two anyon system in a harmonic well as an integral representation of Bessel functions, and as a sum of product of Laguerre's generalized polynomials. This sum is recognized as the spectral representation of the Green's function from which we identify the normalized wave functions and energy spectrum.

It is interesting to note that in the particle-vortex discussion presented by Jackiw [43], he obtained a discrete spectrum of the generator R (our eq. (45)) of conformal transformations, and then, by a rotation in operators space he obtain the continuous spectrum calculated by standard methods to the particle-vortex system. This discrete spectrum is identical to one obtained for anyons in presence of an harmonic regulator. The eigenfunctions (up to a phase factor) and eigenvalues of this operator, eq. (48) in [43], are equivalent to those ones we have obtained for the two anyon system in a harmonic well.

Regarding the Green's functions for the two anyon system in a harmonic well and two anyons in a uniform magnetic field both lead to bound states and we see that they are very similar to each other, although they differ in the energy spectrum. Charged anyons orbiting in a uniform magnetic field are equivalent to anyon bound states in the presence of a harmonic well. Note that wave functions (69) and (80), obtained from those Green's functions, are identical if we identify ω with $\omega_c/2$.

In particular the ground state wave function obtained for two charged particles in a magnetic field, eq. (82), is similar to the two particle wave function used by Laughlin [54] to construct his ansatz for N-particles to describe the quantum Hall effect. In the problem discussed by Laughlin, there is also a coulombic interaction which in general could not be disregarded. Then, he supposed that this interaction is infinitely short ranged and that the Landau levels energy is dominant so $\hbar\omega_c \gg e^2/l$, where l is the magnetic length. Since the particles are separated by some finite length, this allows one to build an ansatz for the many particle ground state wave function as a superposition of single particle wave functions:

$$\psi_{00}^\alpha(r_{p,q}) = e^{i\pi\alpha} \frac{i}{\sqrt{\pi\Gamma(1+\alpha)}} \left(\frac{\mu\omega_c}{2\hbar}\right)^{\frac{1}{2}(1+\alpha)} \prod_{p<q} r_{p,q}^\alpha \exp\left(-\frac{\mu\omega_c}{4\hbar} \sum_{p<q} r_{p,q}^2\right) \quad (83)$$

This is essentially the Laughlin's ansatz for N-particles[54]. Note that the interchange of particle positions adds a phase to the wave function in agreement with [57].

Let us now consider the excited states for the two anyon system in a uniform magnetic field. The wave functions (80), or equivalently (69), represent the excited states of this system. If we follow a similar reasoning as the ones that support (83), as discussed in [49, 50, 54], we can superpose (80) to obtain an ansatz for the many particle excited state wave function:

$$\begin{aligned} \psi_{n,m}^\alpha(r, \phi) &= i e^{i\pi|m-\alpha|} \frac{\sqrt{n!}}{\sqrt{\pi}\sqrt{\Gamma(n+|m-\alpha|+1)}} \left(\frac{\mu\omega_c}{2\hbar}\right)^{\frac{1}{2}(1+|m-\alpha|)} \\ &\times e^{-im\phi} \prod_{i<j} r_{ij}^{|m-\alpha|} L_n^{|m-\alpha|} \left(\frac{\mu\omega_c}{2\hbar} r_{ij}^2\right) \exp\left\{-\frac{\mu\omega_c}{4\hbar} \sum_{i<j} r_{ij}^2\right\}. \end{aligned} \quad (84)$$

This wave function is formally in agreement with the result for many anyons obtained by Dunne et al [46].

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